

# Holography with Gravitational Chern-Simons Term

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## Abstract

The holographic description in the presence of gravitational Chern-Simons term is studied. The modified gravitational equations are integrated by using the Fefferman-Graham expansion and the holographic stress-energy tensor is identified. The stress-energy tensor has both conformal anomaly and gravitational or, if re-formulated in terms of the zweibein, Lorentz anomaly. We comment on the structure of anomalies in two dimensions and show that the two-dimensional stress-energy tensor can be reproduced by integrating the conformal and gravitational anomalies. We study the black hole entropy in theories with a gravitational Chern-Simons term and find that the usual Bekenstein-Hawking entropy is modified. For the BTZ black hole the modification is determined by area of the inner horizon. We show that the total entropy of the BTZ black hole is precisely reproduced in a boundary CFT calculation using the Cardy formula.

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# 1 Introduction

It is amazing how much of physics is encoded in geometry of asymptotically anti-de Sitter (AdS) space-time. This includes the information on the ultra-violet divergences, quantum effective action and the conformal anomaly. The latter is important element in the holographic description and is due to peculiar nature of the asymptotic diffeomorphisms that generate conformal symmetry [1], [2]. The gravitational Einstein-Hilbert action, in which metric is fixed on the boundary, breaks the asymptotic conformal symmetry and is thus the source of the anomalies. This and other related issues were much studied in recent years [3]-[12].

In recent interesting paper [13] Kraus and Larsen have modified the gravitational action in three dimensions by adding the gravitational Chern-Simons term. The resultant theory is known as the topologically massive gravity [14], [15]. Appearance of the Chern-Simons terms is generically predicted in string theory. The gravitational Chern-Simons term, explicitly depending on connection, is gauge invariant only up to some boundary terms. So that its presence necessarily breaks the asymptotic coordinate invariance. The appearance of the gravitational anomaly<sup>2</sup> in the boundary theory is thus should be expected in addition to the already existing conformal anomaly. Alternatively, if the Chern-Simons term is defined in terms of the Lorentz connection, the asymptotic local Lorentz symmetry is broken and the Lorentz anomaly should appear. All these expectations were explicitly confirmed in [13] by looking at how the gravitational action changes under the gauge transformations. On the boundary side the anomaly arises due to different central charge in holomorphic and anti-holomorphic sectors. Such theories were studied some time ago, see in particular [18], [19].

In the present note, inspired by [13], we take a different route to anomalies and show that they follow directly from the bulk gravitational equations. The latter are integrated by expanding the bulk metric in powers of distance from the boundary. This is much in the spirit of [2], [5], [8]. The integration procedure involves the fixing of the boundary data. The data are the boundary metric and the holographic stress-energy tensor. This helps to determine explicitly the structure of stress-energy tensor in terms of the coefficients in the expansion and completely fix the form of the anomalies. The gravitational anomaly we get agrees with the one obtained in [16] and [18]. In the dual picture the holographic stress-energy tensor should be identified with the quasi-local stress-energy tensor which determines the values of mass and angular momentum.

The gravitational Chern-Simons term is eligible gravitational action which produces covariant equations of motion that are, in particular, solved by the BTZ metric. It is therefore interesting question whether the black hole entropy is modified when the Chern-Simons term is included. We study this question and obtain a contribution to the entropy due to the gravitational Chern-Simons term. This contribution depends on the value of the Lorentz connection at the horizon and is, nevertheless, gauge invariant. For the BTZ black hole the entropy due to Chern-Simons term is proportional to the area of inner horizon. This is surprising, taking that in any theory non-linear in Riemann curvature the entropy of BTZ black hole is always, as we argue in the paper, determined by area of the outer horizon. In the theory at hand, the total entropy has dual meaning in terms of the boundary CFT and is precisely reproduced by means of the Cardy formula, as we show.

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<sup>2</sup>For review on the gravitational anomalies see the original works [16], [17].

## 2 Fefferman-Graham expansion and gravitational anomaly

The gravitational theory on three-dimensional space-time is given by the action

$$I_{\text{gr}} = I_{\text{EH}} + I_{\text{CS}} , \quad (2.1)$$

which is sum of ordinary Einstein-Hilbert action (with cosmological constant)

$$I_{\text{gr}} = -\frac{1}{16\pi G_N} \left[ \int_M (R[G] + 2/l^2) + \int_{\partial M} 2K \right] , \quad (2.2)$$

where  $K$  is trace of the second fundamental form of boundary  $\partial M$ , and the gravitational Chern-Simons term

$$I_{\text{CS}} = \frac{\beta}{64\pi G_N} \int_M dx^3 \epsilon^{\mu\nu\alpha} [R_{ab\mu\nu} \omega^{ab}_{,\alpha} + \frac{2}{3} \omega^a_{b,\mu} \omega^b_{c,\nu} \omega^c_{a,\alpha}] . \quad (2.3)$$

Parameter  $l$  in (2.2) sets the AdS scale. To simplify things, on intermediate stage of calculation, we take liberty to use units  $l = 1$  restoring  $l$  explicitly in the final expressions. Parameter  $\beta$  has dimension of length. We use the following definition for the curvature  $R^a_{b\mu\nu} \equiv \partial_\mu \omega^a_{b,\nu} + \omega^a_{c,\mu} \omega^c_{b,\nu} - (\mu \leftrightarrow \nu)$ . The torsion-free Lorentz connection  $\omega^a_b = \omega^a_{b,\mu} dx^\mu$  is determined as usual by equation

$$de^a + \omega^a_b \wedge e^b = 0 , \quad (2.4)$$

where the orthonormal basis  $e^a = h_\mu^a dx^\mu$ ,  $a = 1, 2, 3$  is "square root" of the metric,  $G_{\mu\nu} = h_\mu^a h_\nu^b \eta_{ab}$ . The equation (2.4) can be used to express components of the Lorentz connection in terms of  $h_\mu^a$  and their derivatives<sup>3</sup>

$$\begin{aligned} \omega_{ab,\mu} &= \frac{1}{2} (C_{a\nu\mu} h_b^\nu + C_{b\mu\nu} h_a^\nu - C_{d\alpha\beta} h_a^\alpha h_b^\beta h_\mu^d) , \\ C_{\mu\nu}^a &\equiv \partial_\mu h_\nu^a - \partial_\nu h_\mu^a . \end{aligned} \quad (2.5)$$

The Levi-Civita symbol is determined as  $\epsilon^{\mu\nu\alpha} = h_a^\mu h_b^\nu h_c^\alpha \epsilon^{abc}$ . To complete our brief diving into theory of gravity in orthonormal basis we remind the reader that  $h_\mu^a$  is covariantly constant,  $\nabla_\sigma \epsilon^{\mu\nu\alpha} = 0$ .

The theory described by the action (2.1) is quite well known and belongs to the class of theories with topological mass [14], [15]. A remarkable property of this theory is that it describes a propagating degree of freedom although each term in (2.1) taken separately is topological and thus does not contain local degrees of freedom. Another

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<sup>3</sup>We use the Latin letters  $(a, b, c, d)$  for the inner Lorentz indices and Greek letters  $(\mu, \nu, \alpha, \beta..)$  for the coordinate indices.

interesting property of (2.1) is that the gravitational Chern-Simons tem explicitly breaks the asymptotic coordinate invariance if expressed in terms of the metric connection  $\Gamma^\alpha_{\beta\mu}$  or the asymptotic local Lorentz invariance if the term is written using the Lorentz connection as in (2.3). The violation of the gauge symmetry happens only asymptotically because the variation of the Chern-Simons term under local gauge symmetry generates terms on the boundary of the space-time. This violation thus should be manifest in the boundary theory. Indeed, in [13] this was related to the appearance of the gravitational or Lorentz anomalies in the boundary theory. Such anomalies are natural when  $c_L \neq c_R$  in the boundary conformal field theory. Such theories were studied some time ago, see for example [19]. In the present context, these anomalies are obtained holographically and are encoded in the dynamics of the gravitational field in the bulk. In [13] the anomalies were derived by looking at how the gravitational action (2.1) changes under the gauge symmetries. Here we look at the problem at somewhat different angle. We demonstrate that anomalies show up in the process of the holographic reconstruction of the bulk metric from the boundary data. In the absence of the gravitational Chern-Simons term the bulk metric is uniquely determined once the holographic boundary data, the boundary metric representing the conformal class and the boundary stress tensor, are specified. Usually the boundary stress tensor is not entirely arbitrary. It is covariantly conserved and its trace should reproduce the conformal anomaly. The anomaly itself is completely specified by the boundary metric. The details of this analysis can be found in [3], [5], [8]. The presence of the gravitational Chern-Simons term in the bulk action manifests itself in an interesting way: the boundary stress tensor is no more covariantly conserved. This is how the gravitational anomaly in the boundary theory shows up. In order to see this explicitly we solve the gravitational bulk equations modified by the presence of the Chern-Simons term starting from the boundary and finding the bulk metric as an expansion, well-known in the physics and mathematics literature as the Fefferman-Graham expansion [2].

The gravitational bulk equations obtained by varying the action (2.1) with respect to metric takes the form

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R - G_{\mu\nu} + \beta C_{\mu\nu} = 0 \ , \quad (2.7)$$

where all curvature tensors are determined with respect to the bulk metric  $G_{\mu\nu}$ . The tensor  $C_{\mu\nu}$  is result of the variation of the of the gravitational Chern-Simons term. It is known as the Cotton tensor and takes the form

$$C_{\mu\nu} = \epsilon_\mu^{\alpha\beta} \nabla_\alpha (R_{\beta\nu} - \frac{1}{4}G_{\beta\nu}R) \ . \quad (2.8)$$

Although the Chern-Simons (2.3) is defined in terms of the Lorentz connection which is not gauge invariant object its variation is presented in the covariant and gauge invariant form (2.8). This is just a manifestation of the fact that the "non-invariance" of the Chern-Simons term resides on the boundary and does not appear in the bulk field equations. By virtue of the Bianchi identities this quantity is symmetric, manifestly traceless and identically covariantly conserved,

$$C_{\mu\nu}G^{\mu\nu} = 0 \ , \quad \nabla_\mu C^\mu_\nu = 0 \ , \quad \epsilon_{\alpha\mu\nu}C^{\mu\nu} = 0 \ .$$

Due to these properties we find that solution to the equation (2.7) is space-time with constant Ricci scalar  $R = -6$ . This is exactly what we had when the Chern-Simons term

was not included in the action. In that case we had moreover that  $R_{\mu\nu} = -2G_{\mu\nu}$  and the solution was the constant curvature space. It is no more the case in the presence of the Chern-Simons term and we have

$$R_{\mu\nu} = -2G_{\mu\nu} - \beta C_{\mu\nu} . \quad (2.9)$$

This is that equation which we are going to solve. We start with choosing the bulk metric in the form

$$ds^2 = G_{\mu\nu} dX^\mu dX^\nu = dr^2 + g_{ij}(r, x) dx^i dx^j \quad (2.10)$$

that always can be done by using appropriate normal coordinates. The quantity  $g_{ij}(r, x)$  is induced metric on the hypersurface of constant radial coordinate  $r$ . The following expansion

$$g(r, x) = e^{2r} [g_{(0)} + g_{(2)} e^{-2r} + g_{(4)} e^{-4r} + \dots] \quad (2.11)$$

is assumed so that the metric (2.10) describes asymptotically anti-de Sitter space-time with  $g_{(0)}$  being the metric on its two-dimensional boundary. In the case of pure GR, described by action (2.2), the solution to the gravitational equations contains [5] only these three terms in the expansion (2.11). This is no more true when the Chern-Simons term is turned on and the whole infinite series should be expected in (2.11). The presence of infinite number of terms in the expansion (2.11) generically seems to be related to the presence of local propagating degrees of freedom in the theory.

Now the routine is to insert the expansion (2.11) into the gravitational equations (2.9) and equate the coefficients appearing in front of the same power of  $e^r$  on both sides of the equation. This gives certain constraints on the coefficients  $g_{(2n)}$  appearing in the expansion (2.11) allowing express  $g_{(2n)}$  in terms of the coefficients  $g_{(2k)}$  with  $k < n$ . Generically, in odd dimension  $(d+1)$  there may appear also "logarithmic" term  $h_{(d)} r e^{-dr}$  in (2.11).  $h_{(d)}$  is traceless and covariantly conserved and is local function of boundary metric  $g_{(0)}$ . In  $d = 2$  no such local function of two-dimensional metric exists so that  $h_{(2)}$  identically vanishes (see [5], [8]). When gravitational Chern-Simons term is present same arguments are valid so that no logarithmic term is likely to appear. In any event it would not affect our calculation of the anomalies. Appendices A and B contain details of calculation of the expansion for the Ricci tensor and the Cotton tensor. A good starting point in the analysis is to look at the expansion for the Ricci scalar (A.4). Since the Ricci scalar is constant for solution of equation (2.9) the subleading terms in the expansion for  $R$  should vanish. In the first sub-leading order, as it is seen from (A.4), this gives constraint

$$\text{Tr } g_{(2)} = -\frac{1}{2} R(g_{(0)}) . \quad (2.12)$$

Note that hereafter we define trace with the help of metric  $g_{(0)}$ . Now looking at the equation (2.9) for components  $(\mu\nu) = (ij)$  we find that the leading term vanishes identically and the first subleading term vanishes provided constraint (2.12) is taken into account. Thus, no new constraint on  $g_{(2)}$  appears. Further order terms in the expansion give constraint on higher order terms  $g_{(2k)}$ ,  $k > 2$ . At present we are not interested in those terms. A simple relation of this sort comes from the component  $(\mu\nu) = (rr)$  of equation (2.9)

$$\text{Tr } g_{(4)} = \frac{1}{4} \text{Tr } g_{(2)}^2 - \beta \epsilon^{ij} \nabla_i \nabla_k g_{(2)j}^k \quad (2.13)$$

and indicates that (2.11) is not a "total square" as it happened to be in the case of pure GR [5]. The most important constraint comes from components  $(\mu\nu) = (r, i)$  of the equation (2.9). As it follows from (A.4) and (B.3) we have that

$$-\nabla_j g_{(2)i}^j + \partial_i \text{Tr } g_{(2)} + \beta[\epsilon_i^j(-\nabla_k g_{(2)j}^k + \partial_j \text{Tr } g_{(2)})] = 0 . \quad (2.14)$$

This can be represented in the form

$$\nabla_j t_i^j = \frac{\beta}{2} \epsilon_i^j \partial_j \text{Tr } g_{(2)} , \quad (2.15)$$

where we have introduced symmetric tensor

$$t_{ij} = g_{ij}^{(2)} - g_{ij}^{(0)} \text{Tr } g_{(2)} + \frac{\beta}{2} (\epsilon_i^k g_{(2)kj} + \epsilon_j^k g_{(2)ki}) . \quad (2.16)$$

Equations (2.12) and (2.14) are the only restrictions on coefficient  $g_{ij}^{(2)}$ . Obviously, we can not redefine  $t_{ij}$ , provided it remains symmetric, to include the right hand side of (2.15) so that  $t_{ij}$  would be covariantly conserved. That it is impossible means that in fact we deal with an anomaly. Indeed, the holographic boundary stress tensor defined as

$$T_{ij} = \frac{1}{8\pi G_N} t_{ij} \quad (2.17)$$

has both conformal and gravitational anomalies

$$\text{Tr } T = \frac{l}{16\pi G_N} R , \quad \nabla_j T_i^j = -\frac{\beta}{32\pi G_N} \epsilon_i^j \partial_j R . \quad (2.18)$$

When  $\beta = 0$  the stress tensor defined as in (2.17) agrees with the stress tensor introduced earlier in [4], [5], [8]. In particular, this fixes the coefficient in front of (2.17). The stress tensor (2.17) also agrees with the one suggested in [13]<sup>4</sup>. We see that the conformal anomaly is not affected by the presence of the Chern-Simons term. Taking that conformal symmetry on the boundary of AdS appears as part of bulk diffeomorphisms [7] which are broken by the gravitational Chern-Simons term it is rather non-trivial that the conformal anomaly remains unchanged. On the other hand, the gravitational anomaly (2.18) is entirely due to the Chern-Simons.

The stress tensor  $T_{ij}$  is what usually called the metric stress tensor defined as variation of the action with respect to the metric  $g^{ij}$ . Since we have at our disposal the objects  $h_i^a$  (and their inverse  $h_a^i$ ) which are "square root" of metric,  $g^{ij} = h_a^i h^{aj}$ , we can define what might be called a zweibein stress tensor  $T_i^a$  considering variation of the action with respect to the zweibein  $h_a^i$ . Obviously, we have  $\frac{\delta}{\delta h_a^i} = 2h^{aj} \frac{\delta}{\delta g^{ij}}$  and hence  $T_i^a = 2h^{aj} T_{ij}$ . In Lorentz invariant case antisymmetric part  $T^{[a,b]}$ , where  $T^{ab} = h^{bi} T_i^a$ , vanishes. For the price of loosing the local Lorentz symmetry the zweibein stress tensor  $T_i^a$  can be redefined so that the new stress tensor would be covariantly conserved. Indeed, a new stress tensor

$$\hat{T}_i^a = T_i^a + \frac{\beta l}{16\pi G_N} \epsilon_i^a R , \quad \nabla^j \hat{T}_j^a = 0 \quad (2.19)$$

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<sup>4</sup>Note that our coupling  $\beta$  differs from the one used in [13], exact relation being  $\beta = 32\pi G_N \beta_{\text{KL}}$ .

is covariantly conserved. However, anomaly does not disappear. It reappears as the local Lorentz anomaly. Indeed, we have for the new tensor

$$\epsilon_a^i \hat{T}_i^a = \frac{\beta}{8\pi G_N} R \quad (2.20)$$

that is clear violation of the local Lorentz symmetry under which  $\delta h_a^i = \delta\phi \epsilon_a^b h_b^i$ . This is of course well known: the coordinate invariance can be restored for the price of loosing the local Lorentz invariance.

It is of obvious interest to analyze the gravitational anomaly which may appear in higher dimension  $d = 4n + 2$  when gravitational Chern-Simons term<sup>5</sup> is added to the  $(d + 1)$ -dimensional Einstein-Hilbert action. This is currently under investigation.

### 3 Remarks on anomalies in two dimensions

#### 3.1 Local counterterms, conformal and Lorentz anomalies

Once the Lorentz symmetry is broken anyway it is allowed to add local counterterms to the boundary action that are not Lorentz invariant. Appropriate counterterms depend on the zweibein  $h_i^a$ ,  $a = 1, 2$ , rather than on the metric. It is interesting that by adding such local counterterms we can shift the value of the conformal anomaly—the possibility which we did not have when dealt only with metric. The counterterm of this sort was suggested in [20]

$$I_{\text{ct}} = \frac{1}{4} \int d^2x \, h \, C_{ij}^a C_a^{ij} \quad , \quad (3.1)$$

where  $C_{ij}^a = \partial_i h_j^a - \partial_j h_i^a$  is the anholonomy object for the zweibein  $h_i^a$ ,  $a = 1, 2$  on the boundary and  $h = \det h_i^a$ . Notice, that this term added on the regulated boundary (at fixed value of radial coordinate  $r$ ) is finite in the limit when  $r$  is infinite.

The Lorentz group in two dimensions is abelian so that the Lorentz connection has only one component

$$\omega_{b,i}^a = \epsilon_b^a \omega_i \quad , \quad \omega_i = \frac{1}{2} \omega_{ab,i} \epsilon^{ab} \quad .$$

Under local Lorentz and conformal transformation  $\delta h_i^a = \delta\sigma h_i^a + \delta\phi \epsilon_a^b h_b^i$  the Lorentz connection transforms as

$$\delta\omega_i = \partial_i \delta\phi + \epsilon_i^j \partial_j \delta\sigma \quad . \quad (3.2)$$

The counterterm (3.1) changes as follows

$$\delta I_{\text{ct}} = \frac{1}{2} \int d^2x \, h \, [\delta\sigma R + \delta\phi K] \quad . \quad (3.3)$$

$R$  is the two-dimensional Ricci scalar which can be expressed in terms of the Hodge dual to the Lorentz connection one-form

$$R = 2\nabla_i(\tilde{\omega}^i) \quad , \quad \tilde{\omega}^i = \epsilon^i_j \omega^j \quad .$$

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<sup>5</sup>We mean here the Chern-Simons term for the local Lorentz group  $SO(1, d)$ . Other possible Chern-Simons terms, for instance for group  $SO(2, d)$ , do not seem to produce gravitational anomaly on the boundary of AdS [12].

The quantity  $K$  that appears in (3.3) has similar expression in terms of the Lorentz connection itself

$$K = 2\nabla_i(\omega^i) . \quad (3.4)$$

It is invariant under conformal transformations and changes under the local Lorentz transformations. There is certain similarity between  $R$  and  $K$  well discussed in [20]. It is important that both  $R$  and  $K$  may appear in conformal and/or Lorentz anomaly. Obviously, adding (3.1) with appropriate coefficient to the boundary effective action we can always shift the value of the conformal anomaly and even remove it completely. As a price for that the quantity  $K$  would appear in the Lorentz anomaly.

### 3.2 Stress-energy tensor from anomalies

Two-dimensional black hole can be put on the boundary of three-dimensional anti-de Sitter, the two-dimensional Hawking effects then would be encoded in the bulk three-dimensional geometry [5]. It is well-known that conformal anomaly plays important role in two dimensions and eventually is responsible for the Hawking effect. Important element in this demonstration [21] is the observation that the conformal anomaly can be integrated to determine the covariantly conserved stress-energy tensor. In this subsection we analyze whether this is still true when the gravitational anomaly is present. Thus, we would like to see whether the equations

$$T_{ij}g^{ij} = aR , \quad \nabla_j T_i^j = -b\epsilon_i^j \partial_j R , \quad (3.5)$$

where  $a$  and  $b$  are some constants, can be integrated and determine the stress energy tensor  $T_{ij}$ . Constants  $a$  and  $b$  can be further related to the central charge in left- and right-moving sectors of two-dimensional theory as we discuss it in section 5. The exact relation is  $a = \frac{c_+}{24\pi}$  and  $b = \frac{c_-}{48\pi}$ ,  $c_\pm = (c_L \pm c_R)/2$ .

We start with two-dimensional static metric in the Schwarzschild like form

$$ds^2 = -g(x)dt^2 + \frac{1}{g(x)}dx^2 , \quad (3.6)$$

where  $g(x)$  is some function of the spatial coordinate  $x$ . The only non-vanishing Christoffel symbols for this metric are

$$\Gamma_{tx}^t = \frac{g'}{2g} , \quad \Gamma_{tt}^x = \frac{gg'}{2} ,$$

where  $g' = \partial_x g$ , and the scalar curvature takes the simple form  $R = -g''(x)$ . Assuming that components of the stress tensor  $T_{ij}$  do not depend on time  $t$  we get that equations (3.5) are equivalent to a set of differential equations

$$\begin{aligned} T_x^x + T_t^t &= -ag'' \\ \partial_x T_x^x + \frac{g'}{2g}(T_x^x - T_t^t) &= 0 \\ \partial_x T_t^x &= bgg''' . \end{aligned} \quad (3.7)$$

We chose orientation in which  $\epsilon^{tx} = +1$  when derived (3.7). These equations can be solved and the solution reads

$$\begin{aligned} T_t^t &= a(-g'' + \frac{g'^2}{4g} + \frac{C_1}{g}) \\ T_{xt} &= b(g'' - \frac{g'^2}{2g} + \frac{C_2}{g}) \end{aligned} \quad , \quad (3.8)$$

where  $C_1$  and  $C_2$  are integration constants. The Hawking temperature of two-dimensional black hole is  $T_H = g'(x_+)/4\pi$ , where  $x_+$  is location of horizon defined as simple root of function  $g(x_+) = 0$ . The condition of regularity (see for instance [22]) of  $T_t^t$  and  $T_{xt}$  at horizon fixes the constants  $C_1 = -\frac{1}{4}g'^2(x_+)$  and  $C_2 = \frac{1}{2}g'^2(x_+)$ . The stress tensor thus can be uniquely reproduced from the anomaly equations (3.5). We see that the gravitational anomaly shows up only in the component  $T_{xt}$  which is now non-vanishing and proportional to  $b$ . If the two-dimensional space-time is asymptotically flat, i.e.  $g(x) \rightarrow 1$  when  $x \rightarrow \infty$ , then (3.8) describes at infinity a non-vanishing flow

$$T_{tt} = c_+ \frac{\pi}{6} T_H^2, \quad T_{xt} = c_- \frac{\pi}{6} T_H^2 \quad (3.9)$$

due to the Hawking particles radiated by black hole.

## 4 Black hole entropy from gravitational Chern-Simons term

The gravitational Chern-Simons term is a legitimate action for gravitational field. It produces covariant field equations which might have sensible solutions<sup>6</sup>. In particular, the constant curvature space-time is always solution of these equations and remains to be a solution when the gravitational dynamics is governed, as in (2.1), by the sum of Einstein-Hilbert action and the Chern-Simons term. The BTZ black hole is thus a solution to the equations (2.7), as was first noted in [23]. On the other hand, it is well known that the expression for the Bekenstein-Hawking entropy is modified if gravitational action is non-linear or even non-local function of curvature. In general the entropy is not just a quarter of horizon area but depends also on the way horizon is embedded in the space-time. It is thus interesting question whether the gravitational Chern-Simons (2.3) leads to any modifications of the entropy. The tricky point here is that the Chern-Simons is defined with respect to the Lorentz connection so that one might worry whether the corresponding entropy is gauge invariant. In this section we analyze this issue.

There are various ways to compute the entropy for a given gravitational action. The most popular is the Wald's Noether charge method [24]. It is however a on-shell method which is valid on the equations of motion. Below we use another method which is universal and does not rely on the equations of motion. This is the method of conical singularity [25], [22], [26]. The idea is to allow black hole to have temperature different from the Hawking one. In the Euclidean description this leads to the appearance of deficit angle  $\delta = 2\pi(1-\alpha)$ ,  $\alpha = T_H/T$  at horizon  $\Sigma$ . The geometry of manifolds with conical singularities

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<sup>6</sup>The corresponding equations of motion  $C_{\mu\nu} = 0$  are satisfied for any conformally flat 3d metric.

was analyzed in detail in [26]. In particular, it was found that components of the Riemann tensor contain a singular, delta-function like, part

$$R^{\alpha\beta}_{\mu\nu} = (R^{\alpha\beta}_{\mu\nu})_{\text{reg}} + 2\pi(1-\alpha)[(n^\alpha n_\mu)(n^\beta n_\nu) - (n^\alpha n_\nu)(n^\beta n_\mu)]\delta_\Sigma , \quad (4.1)$$

where  $(R^{\alpha\beta}_{\mu\nu})_{\text{reg}}$  is non-singular part of the curvature;  $(n^\alpha n_\mu) = n_1^\alpha n_\mu^1 + n_2^\alpha n_\mu^2$ ,  $n_1$  and  $n_2$  is pair of vectors normal to  $\Sigma$  and orthogonal to each other. Obtained originally in [26] for static non-rotating metric, this formula was later shown in [27] to be correct in the case of stationary metric.

Taking into account (4.1), the gravitational action in question is now function of  $\alpha$ . The entropy then is defined as

$$S = (\alpha \frac{\partial}{\partial \alpha} - 1)|_{\alpha=1} I_{\text{gr}}(\alpha) . \quad (4.2)$$

Applying this formula to the Chern-Simons term (2.3) we get<sup>7</sup>

$$S_{\text{CS}} = -\frac{\beta}{8G_N} \int_{\Sigma} \omega_{ab,\sigma} h_\alpha^a h_\beta^b \epsilon^{\mu\nu\sigma} (n^\alpha n_\mu)(n^\beta n_\nu) \quad (4.3)$$

for the entropy. Note that indices  $a, b$  run values from 1 to 3. In the case of (2+1)-dimensional black hole the horizon  $\Sigma$  is circle. Suppose  $\varphi$  is the angular coordinate on  $\Sigma$  then vector  $\partial_\varphi$  is orthogonal to  $n_1$  and  $n_2$ . We assume that vector  $\partial_\varphi$  together with vector  $\partial_\tau$  form a pair of Killing vectors at horizon. (Outside horizon the Killing vectors are linear combinations of these two vectors.) It follows that the integrand in (4.3) is non-vanishing only if index  $\sigma = \varphi$ . Introducing  $\hat{\epsilon}^{\alpha\beta} = \epsilon^{\mu\nu\varphi} (n^\alpha n_\mu)(n^\beta n_\nu)$  expression (4.3) can be re-written as

$$S_{\text{CS}} = -\frac{\beta}{8G_N} \int_{\Sigma} \omega_{ab,\varphi} h_\alpha^a h_\beta^b \hat{\epsilon}^{\alpha\beta} . \quad (4.4)$$

As far as we are aware, the result (4.3), (4.4) for the Chern-Simons entropy is new. Under local Lorentz transformations parameterized by  $\Omega_{ab}$  the expression (4.4) changes as

$$\delta S_{\text{CS}} = -\frac{\beta}{8G_N} \int_{\Sigma} [\partial_\varphi(\Omega_{ab}) \hat{\epsilon}^{ab}] \gamma d\varphi , \quad (4.5)$$

where  $\gamma$  is induced measure on  $\Sigma$ . Since  $\partial_\varphi$  is Killing vector the quantity  $\hat{\epsilon}^{ab} = h_\alpha^a h_\beta^b \hat{\epsilon}^{\alpha\beta}$ , being considered on  $\Sigma$ , does not depend on  $\varphi$ . Therefore, integrating by parts in (4.5) we find that  $\delta S_{\text{CS}} = 0$ , i.e. entropy (4.4) is Lorentz invariant in spite the fact that the Lorentz connection enters explicitly in (4.4).

## The BTZ black hole

The BTZ black hole is important, and in fact the only one known, example of black hole in three dimensions<sup>8</sup>. Therefore it is interesting to see how our formulas work in this case.

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<sup>7</sup>For spin connection one has that  $\omega = \omega_{\text{reg}} + \omega_{\text{sing}}$  so that  $R_{\text{reg}} = d\omega_{\text{reg}} + \dots$  is regular and  $R_{\text{sing}} = d\omega_{\text{sing}}$  is the singular part in (4.1). Therefore, schematically, one has  $\int \omega R = \int \omega_{\text{reg}} R_{\text{reg}} + 2 \int \omega_{\text{reg}} R_{\text{sing}}$ . This gives extra factor of 2 when (4.1) is applied to action (2.3).

<sup>8</sup>For a recent review on the BTZ black hole, conformal field theory and three-dimensional gravity see [30].

The orthonormal basis  $e^a = h_\mu^a dx^\mu$  for the BTZ metric is

$$e^1 = \sqrt{f(r)} dt, \quad e^2 = \frac{1}{\sqrt{f(r)}} dr, \quad e^3 = r(d\varphi + N(r)dt), \quad (4.6)$$

where

$$f(r) = \frac{r^2}{l^2} - \frac{j^2}{r^2} - m = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2}, \quad N(r) = -\frac{j}{r^2}. \quad (4.7)$$

We have that

$$m = \frac{r_+^2 + r_-^2}{l^2}, \quad j = \frac{r_+ r_-}{l}. \quad (4.8)$$

Here we work in the Lorentzian signature. The analytic continuation to the Euclidean signature was analyzed in [28] and [29]. The vectors orthogonal to horizon are

$$n_1 = \frac{1}{\sqrt{f}}(\partial_t - N\partial_\varphi), \quad n_2 = \sqrt{f}\partial_r \quad (4.9)$$

so that we have that

$$\begin{aligned} (n^t n_t) &= (n^r n_r) = 1, \quad (n^\phi n_t) = -N(r_+) \\ \hat{\epsilon}^{tr} &= \frac{1}{r_+}, \quad \hat{\epsilon}^{\varphi r} = \frac{N(r_+)}{r_+} \\ \hat{\epsilon}^{12} &= \frac{1}{r_+}, \quad \hat{\epsilon}^{13} = \hat{\epsilon}^{23} = 0. \end{aligned} \quad (4.10)$$

Taking into account that measure  $\gamma = r_+$  on  $\Sigma$  we find that the expression for entropy takes a simple form

$$S_{\text{CS}} = -\frac{\beta}{4G_N} \int_0^{2\pi} \omega_{12,\varphi} d\varphi. \quad (4.11)$$

Explicit calculation, making use of eq. (2.5), shows that

$$\omega_{12,\varphi} = \frac{j}{r_+}. \quad (4.12)$$

The contribution to the entropy due to the Chern-Simons term

$$S_{\text{CS}} = -\frac{\beta}{4G_N} \frac{2\pi r_-}{l} \quad (4.13)$$

is thus proportional to the area  $2\pi r_-$  of the inner horizon. That's a curious property of the gravitational Chern-Simons term. Its entropy is apparently due to degrees of freedom at inner horizon rather than at the horizon which may be seen by an external observer. We will comment on this interesting feature later in the paper. Summing the contributions to the entropy that come from each term in the gravitational action (2.1) the total entropy of BTZ black hole is

$$S_{\text{BH}} = \frac{2\pi r_+}{4G_N} - \frac{\beta}{l} \frac{2\pi r_-}{4G_N}. \quad (4.14)$$

Depending on the sign of  $\beta$  the contribution of the Chern-Simons term to the entropy may be negative. This is not a problem as soon as the total entropy (4.14) is positive. This imposes certain bound on possible values of  $\beta$ . We discuss this in the next section.

## 5 The boundary CFT calculation

In this section we use the representation when the Lorentz symmetry is broken but the theory is diffeomorphism invariant, so that the stress-energy tensor of the dual theory is covariantly conserved. The boundary CFT in question is characterized by different values of the central charge for holomorphic and anti-holomorphic fields. The zweibein stress tensor of the theory has both conformal and Lorentz anomalies. Summarizing our analysis in section 2 we have

$$h_a^i \hat{T}_i^a = \frac{c_+}{12\pi} R, \quad \epsilon_a^i \hat{T}_i^a = \frac{c_-}{12\pi} R, \quad (5.1)$$

where  $c_{\pm} = \frac{1}{2}(c_L \pm c_R)$  and  $c_L$  ( $c_R$ ) is central charge for the left-(right-) moving sector. Expressions (2.18) and (2.20) give precise values for the central charge in each sector<sup>9</sup>

$$c_L = \frac{3(l + \beta)}{2G_N}, \quad c_R = \frac{3(l - \beta)}{2G_N}. \quad (5.2)$$

The BTZ black hole corresponds to the sector in the boundary CFT characterized by conformal weights [1]

$$h_L = \frac{Ml - J}{2}, \quad h_R = \frac{Ml + J}{2} \quad (5.3)$$

that are determined by mass  $M$  and angular momentum  $J$  of black hole. These two parameters are the integrals

$$M = l \int_0^{2\pi} d\varphi T_{tt}, \quad J = -l \int_0^{2\pi} d\varphi T_{t\varphi}$$

of the components of the stress tensor defined in (2.17), (2.16). The coefficients in the Fefferman-Graham expansion of BTZ metric are collected in Appendix C. These are needed for computing the stress tensor using (2.17), (2.16). We then get

$$M = M_0 - \frac{\beta}{l^2} J_0, \quad J = J_0 - \beta M_0, \quad (5.4)$$

where quantities

$$M_0 = \frac{r_+^2 + r_-^2}{8G_N l^2}, \quad J_0 = \frac{r_+ r_-}{4G_N l} \quad (5.5)$$

are values of mass and angular momentum in the absence of the Chern-Simons term. The shift (5.4) has been recently found in [13]. In fact it was known for some time (see [31], [32]) that mass and angular momentum in topologically massive gravity are linear combinations of mass and angular momentum obtained in pure GR.

The entropy in the boundary CFT is computed by the Cardy formula

$$S_{\text{CFT}} = 2\pi \left( \sqrt{\frac{c_L h_L}{6}} + \sqrt{\frac{c_R h_R}{6}} \right). \quad (5.6)$$

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<sup>9</sup>Similar shift in central charge was observed in [31] within the Brown-Henneaux approach. I thank S. Carlip for drawing my attention to this reference.

Plugging here the known values for the central charge and conformal weight in each sector we find

$$S_{\text{CFT}} = \frac{2\pi r_+}{4G_N} - \frac{\beta}{l} \frac{2\pi r_-}{4G_N} \quad (5.7)$$

that is in perfect agreement with the black hole entropy (4.14) computed in previous section. When the bulk theory is pure GR the agreement is well known [33]. The gravitational and CFT entropies still agree when the gravitational Chern-Simons term is added in the bulk, as we have just shown. Notice, that this bulk theory is much richer than GR since it now contains propagating degrees of freedom.

Apparently, large values (of any sign) of the coupling  $\beta$  are not allowed in the theory. There are two obvious signals of instability for large  $\beta$ . Central charge in either of two sectors may become negative. Also, entropy becomes negative when  $\beta$  is "too large". These bad things do not happen if parameter  $\beta$  is within the range

$$|\beta| \leq l \quad . \quad (5.8)$$

This "stability bound" guarantees that both the bulk theory with the gravitational Chern-Simons term and the boundary CFT with  $c_L \neq c_R$  are well-defined.

## 6 Does the Chern-Simons term look deep into black hole?

In theories of gravity involving higher powers of Riemann tensor the black hole entropy is no longer the usual  $A/4G_N$  and is always modified. This is well known and quite well understood. We refer the reader to [34] for the Noether charge calculation and to [26] for the calculation that uses the conical singularity method. For black holes arising in string theory this issue was much studied, see review in [35]. For BTZ black hole and higher-dimensional black holes that reduce to BTZ, this issue was studied in [36] and recently in [37]. We would like to discuss here some interesting peculiarities of higher curvature modifications of the entropy of BTZ black hole.

The general action of local theory of gravity with higher derivatives can be represented as a power series in Riemann curvature. This in fact is true also for a non-local theory however each term in such expansion then would contain non-local factors. Keeping theory local the quadratic term in our action would be something like this

$$W = \int \left( \frac{a_1}{24\pi} R^2 + \frac{a_2}{16\pi} R_{\mu\nu}^2 + \frac{a_3}{16\pi} R_{\alpha\beta\mu\nu}^2 \right) \quad . \quad (6.1)$$

The corresponding contribution (see [34] and [26] for more detail) to the entropy is

$$S = - \int_{\Sigma} \left( \frac{a_1}{3} R + \frac{a_2}{4} R_{\mu\nu} (n^\mu n^\nu) + \frac{a_3}{2} R_{\mu\nu\alpha\beta} (n^\mu n^\alpha) (n^\nu n_\beta) \right) \quad (6.2)$$

as can be easily obtained using method outlined in section 4. Applying this to BTZ black hole we notice that the BTZ metric is locally AdS and hence the Riemann tensor factorizes  $R_{\alpha\beta\mu\nu} = \frac{1}{l^2} (G_{\beta\mu} G_{\alpha\nu} - G_{\alpha\mu} G_{\beta\nu})$ . This factorization and that vectors  $n_1$  and  $n_2$  are orthonormal lead to an interesting conclusion that nothing in the integrand in (6.2)

depends on the parameters of black hole. Those parameters enter (6.2) only via area of  $\Sigma$ , i.e. via  $r_+$ ,

$$S = (a_1 + a_2 + a_3) \frac{2\pi r_+}{l} . \quad (6.3)$$

Obviously this property remains in place when higher powers of curvature are included in the action. In fact we can state that any local theory of gravity that is non-linear in curvature results in the entropy which takes the form

$$S_{\text{non}} = \mu(a_i, l) \frac{2\pi r_+}{l} , \quad (6.4)$$

where  $\mu(a_i, l)$  is some function of higher curvature couplings  $a_i$  and the AdS scale  $l$  but not of the parameters of black hole. Similar result was recently derived in [37]. The higher derivative theory of gravity, provided it is formulated in terms of gauge invariant objects, i.e. the Riemann tensor, thus sees only the radius  $r_+$  of outer horizon of BTZ black hole and leaves  $r_-$  unnoticed<sup>10</sup>.

The gravitational Chern-Simons term, as we have seen in section 5, shows radically different behavior. Its entropy is proportional to the area  $2\pi r_-$  of inner horizon so that it is  $r_+$  that is now unnoticed. This is despite the fact that the entropy is actually given by integral (4.3) over outer horizon. This is an interesting feature of the gravitational Chern-Simons term that it seems to see the interior of the black hole. The Lorentz connection apparently does the trick. Most dramatically this feature manifests itself when the gravitational action contains the Chern-Simons term only. The BTZ metric is still a solution to the field equation. Its temperature, mass and angular momentum are non-vanishing and hence, thermodynamically, there must be some entropy and that entropy is precisely  $S_{\text{cs}}$  defined in (4.13) and determined by the area of inner horizon. Notice, that it constitutes entire entropy of black hole in this case! This observation poses interesting questions that may be challenging to our present understanding of the black hole entropy. The obvious one is whether there should be some degrees of freedom associated with the inner (rather than with the outer) horizon which would be responsible for this entropy? We leave this and other questions for the future.

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<sup>10</sup>In general, this may be different in the case of non-local theory of gravity. Such a theory may produce logarithmic terms in the entropy and both  $\ln r_+$  and  $\ln r_-$  are a priori possible. The concrete calculation in [29] however shows that to the leading order such entropy is determined by  $r_+$  only,  $r_-$  appearing in the subleading terms.

# Appendix

## A Curvature components and their expansion

Components of the Riemann tensor are

$$\begin{aligned} R^r_{irj} &= \frac{1}{2}[-g'' + \frac{1}{2}g'g^{-1}g']_{ij} \\ R^r_{ikj} &= -\frac{1}{2}[\nabla_k g'_{ij} - \nabla_j g'_{ik}] \\ R^l_{ikj} &= R^l_{ikj}(g) - \frac{1}{4}g'_{ij}g^{ln}g'_{nk} + \frac{1}{4}g'_{ik}g^{ln}g'_{nj} , \end{aligned} \quad (\text{A.1})$$

where  $g' \equiv \partial_r g$ . Components of Ricci tensor are

$$\begin{aligned} R_{ij} &= R_{ij}(g) - \frac{1}{2}g''_{ij} - \frac{1}{4}g'_{ij}\text{Tr}(g^{-1}g') + \frac{1}{2}(g'g^{-1}g')_{ij} \\ R_{ri} &= \frac{1}{2}[\nabla^k(g^{-1}g')_{ki} - \nabla_i\text{Tr}(g^{-1}g')] \\ R_{rr} &= -\frac{1}{2}\text{Tr}(g^{-1}g'') + \frac{1}{4}\text{Tr}(g^{-1}g'g^{-1}g') \end{aligned} \quad (\text{A.2})$$

and the Ricci scalar is

$$R = R(g) - \text{Tr}(g^{-1}g'') - \frac{1}{4}[\text{Tr}(g^{-1}g')]^2 + \frac{3}{4}\text{Tr}(g^{-1}g'g^{-1}g') . \quad (\text{A.3})$$

The leading terms in the Fefferman-Graham expansion of the curvature tensors are

$$\begin{aligned} R_{ri} &= [-\nabla_n g_{(2)i}^n + \partial_i \text{Tr} g_{(2)}]e^{-2r} + \dots \\ R^k_i &= -2\delta_i^k + [R_i^k(g_{(0)}) + \delta_i^k \text{Tr} g_{(2)}]e^{-2r} + \dots \\ R_{rr} &= -2 + [-4\text{Tr} g_{(4)} + \text{Tr} g_{(2)}^2]e^{-4r} + \dots \\ R &= -6 + [R(g_{(0)}) + 2\text{Tr} g_{(2)}]e^{-2r} + \dots \end{aligned} \quad (\text{A.4})$$

For the constant curvature  $R = -6$  metric we have a constraint

$$\text{Tr} g_{(2)} = -\frac{1}{2}R_{(0)} . \quad (\text{A.5})$$

## B Components of the Cotton tensor and their expansion

In space-time with constant Ricci scalar  $R = -6$  the Cotton tensor is defined as

$$C_{\alpha\beta} = \epsilon_{\alpha}^{\mu\nu}\nabla_{\mu}R_{\nu\beta} . \quad (\text{B.1})$$

For the Levi-Civita symbol we have that  $\epsilon^{rij} = \epsilon^{ij}$  where  $\epsilon^{ij}$  is defined for the 2d metric  $g_{ij}(r, x)$ .

In terms of  $g_{ij}(r, x)$  we get for the components of (B.1)

$$\begin{aligned} C_{ri} &= -\epsilon_n^k \nabla_k R_i^n + \frac{1}{2} \epsilon^{kn} g'_{ki} R_{rn} \\ C_{rr} &= \epsilon^{ij} [\nabla_i R_{rj} - \frac{1}{2} (g^{-1} g')_i^k R_{kj}] \\ C_{ij} &= -\epsilon_i^k [\partial_r R_{kj} - \frac{1}{2} (g^{-1} g')_j^n R_{kn} - \nabla_k R_{rj} - \frac{1}{2} g'_{kj} R_{rr}] . \end{aligned} \quad (\text{B.2})$$

Taking into account the constraint (A.5) we find the following expansion for the components of the Cotton tensor

$$\begin{aligned} C_{ri} &= \epsilon_i^j [-\nabla_k g_{(2)j}^k + \partial_j \text{Tr } g_{(2)}] e^{-2r} + \dots \\ C_{rr} &= -\epsilon^{ij} \nabla_i \nabla_k g_{(2)j}^k e^{-4r} + \dots \\ C_{ij} &= 0 + O(e^{-2r}) , \end{aligned} \quad (\text{B.3})$$

where the leading term (of order  $e^{0r}$ ) in the expansion of  $C_{ij}$  vanishes due to constraint (A.5).

## C The BTZ metric in normal coordinates

The BTZ metric can be brought to the normal coordinates in the form (2.10) as follows

$$\begin{aligned} ds^2 &= dr^2 - \left( \frac{r_+^2}{l^2} \sinh^2 \frac{r}{l} - \frac{r_-^2}{l^2} \cosh^2 \frac{r}{l} \right) dt^2 \\ &\quad + \left( \frac{r_+^2}{l^2} \cosh^2 \frac{r}{l} - \frac{r_-^2}{l^2} \sinh^2 \frac{r}{l} \right) d\varphi^2 - \frac{2r_+ r_-}{l} dt d\varphi , \end{aligned} \quad (\text{C.1})$$

where  $r_+$  ( $r_-$ ) is radius of outer (inner) horizon. The coefficients in the expansion (2.11) of this metric are

$$\begin{aligned} g_{tt}^{(0)} &= -\frac{1}{l^2} g_{\varphi\varphi}^{(0)} = -\frac{(r_+^2 - r_-^2)}{4l^2} \\ g_{tt}^{(2)} &= \frac{1}{l^2} g_{\varphi\varphi}^{(2)} = \frac{(r_+^2 + r_-^2)}{2l^2} , \quad g_{t\varphi}^{(2)} = -\frac{r_+ r_-}{l} . \end{aligned} \quad (\text{C.2})$$

We choose orientation in which  $\epsilon_t^\varphi = -1/l$ .

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